Int J Theor Phys (2009) 48: 2319–2327
DOI 10.1007/s10773-009-0021-2

Thermal Effect for the Mesoscopic LC Circuits Including Complicated Coupling by Virtue of GHFT

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Received: 18 February 2009 / Accepted: 15 April 2009 / Published online: 22 April 2009

Abstract The Hamilton operator for the mesoscopic LC circuits including complicated coupling is given. Because the practical circuits are inevitably influenced by the external environment, the effect of temperature on the quantum effects for the mesoscopic circuit must be taken into account. Then we discuss the thermal effect of the system by simulating it with the biphotonic process and the contribution of each process to the energy ensemble average by virtue of the generalized Hellmann-Feynman Theorem (GHFT).

Keywords Mesoscopic · Thermal effect · GHFT · Characteristics

1 Introduction

In recent years, physicists are fascinated by the potential use of quantum computation. For the purpose of embodying the quantum computation, some systems have been investigated [1–5]. Among them, the solid-state proposals such as quantum dots and Josephson tunnel junctions have become attractive candidates because of the advantages of its designing flexibility and its parameters being adjusted continuously over a wide range [6], which is promising for the circuit large scale integration. Certainly, in order to realize practical quantum circuit integration, it is inevitable to connect Josephson junctions with other circuit cells such as resistors, inductors and capacitors. Therefore, it is still significant to investigate the quantum effects of mesoscopic circuits. A single LC (inductance-capacitance) nondissipative mesoscopic circuit is a fundamental cell in mesoscopic circuits and its quantization and quantum effects at the absolute temperature were first discussed by Louisell [7]. For practical circuits are inevitably influenced by the external environment,

Project supported by NSFC (No. 10574060), the Natural Science Foundation of Shandong Province (No. Y2008A23) and the Science Foundation of Liaocheng University (No. X071045).

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the effect of temperature on quantum effects for the mesoscopic circuits must be taken into account. In the past few years, the quantum effects at finite temperature for mesoscopic circuits have attracted attention [8–11]. The methods usually employed were mainly based on the thermal field dynamics (TFD) [12]. Takahashi and Umezawa converted the ensemble average to the pure state average by introducing a virtual space. The expense of this convenience is doubling the Hilbert space. Fan et al. constructed the generalized Hellmann-Feynman Theorem (GHFT) [13], which is convenient to calculate expectation values in a mixed state. In this work, we shall discuss the thermal effects for the mesoscopic LC circuits including complicated coupling (see Fig. 1). Our procedures are as follows, Firstly, for convenience, we introduce GHFT. Secondly, we quantize the mesoscopic LC circuit including complicated coupling. Finally, we discuss the thermal effect by employing GHFT.

2 The Generalized Hellmann-Feynman Theorem

The construction of GHFT is corresponding to the Hellmann-Feynman Theorem (HFT) [14, 15], which has wide applications in quantum mechanical expectation value calculations and can be employed to analyse some complicated problems without doing tedious evaluations. But the expectation values by HFT is from the pure state consideration. When the state describing the system is a mixed state, it is necessary to extend HFT to the case of a mixed state.

Supposing $\hat{\mathcal{H}}(\lambda)$ is the Hamilton operator of a parameterized system, and it satisfies the following eigen-equation

$$\hat{\mathcal{H}}(\lambda)\psi_n(\lambda) = E_n(\lambda)\psi_n(\lambda). \tag{1}$$

By virtue of HFT, we have

$$\frac{\partial E_n}{\partial \lambda} = \langle \psi_n | \frac{\partial \hat{\mathcal{H}}}{\partial \lambda} | \psi_n \rangle. \tag{2}$$

If the state is mixed state, we have to deal with the problem by density operator ρ

$$\rho \equiv \sum_n e^{-\beta \hat{\mathcal{H}}_n} |\psi_n\rangle\langle\psi_n|, \tag{3}$$

where $\beta = (kT)^{-1}$, k is the Boltzmann constant and T, temperature. In this case the average energy $\langle \hat{\mathcal{H}}(\lambda)\rangle_e$ (here the subscript e denotes ensemble average) is

$$\langle \hat{\mathcal{H}}(\lambda)\rangle_e = \frac{\mathrm{Tr}(\rho \hat{\mathcal{H}}(\lambda))}{Z(\lambda)} = \frac{1}{Z(\lambda)} \sum_n e^{-\beta E_n(\lambda)} E_n(\lambda) = \bar{E}(\lambda),\ Z(\lambda) = \mathrm{Tr}(\rho). \tag{4}$$

Differentiating $\bar{E}(\lambda)$ by the parameter λ can lead to

$$\begin{aligned}\frac{\partial \bar{E}(\lambda)}{\partial \lambda} &= \frac{1}{Z^2(\lambda)} \Bigg\{ Z(\lambda) \sum_n e^{-\beta E_n(\lambda)} [-\beta E_n(\lambda) + 1] \frac{\partial E_n(\lambda)}{\partial \lambda} \\ &\quad - \left[\sum_n e^{-\beta E_n(\lambda)} E_n(\lambda)\right]\left[\sum_n e^{-\beta E_n(\lambda)} \frac{\partial E_n(\lambda)}{\partial \lambda}(-\beta)\right]\Bigg\} \\ &= \frac{1}{Z(\lambda)} \left\{ \sum_n e^{-\beta E_n(\lambda)} [-\beta E_n(\lambda) + \beta \bar{E}(\lambda) + 1] \frac{\partial E_n(\lambda)}{\partial \lambda} \right\}. \end{aligned} \tag{5}$$

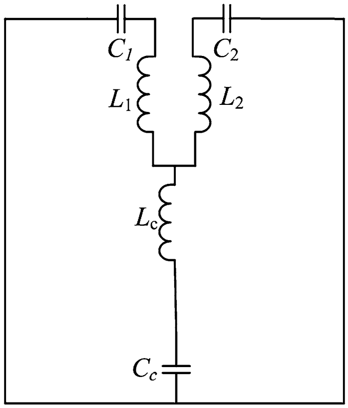

Fig. 1 Mesoscopic LC circuits including complicated coupling

Substituting (4) into (5) we have

$$\frac{\partial\langle\hat{\mathcal{H}}(\lambda)\rangle_e}{\partial\lambda}=\frac{\partial\bar{E}(\lambda)}{\partial\lambda}=\left\langle[1+\beta\bar{E}(\lambda)-\beta\hat{\mathcal{H}}(\lambda)]\frac{\partial\hat{\mathcal{H}}(\lambda)}{\partial\lambda}\right\rangle_e. \tag{6}$$

This is the generalization of the well-known Hellmann-Feynman Theorem (HFT), named generalized Hellmann-Feynman Theorem (GHFT). When $\hat{\mathcal{H}}(\lambda)$ is independent of β, (6) can be rewritten as

$$\begin{aligned}\frac{\partial\bar{E}(\lambda)}{\partial\lambda}&=[1+\beta\bar{E}(\lambda)]\left\langle\frac{\partial\hat{\mathcal{H}}(\lambda)}{\partial\lambda}\right\rangle_e+\beta\frac{\partial}{\partial\beta}\left\langle\frac{\partial\hat{\mathcal{H}}(\lambda)}{\partial\lambda}\right\rangle_e-\beta\bar{E}(\lambda)\left\langle\frac{\partial\hat{\mathcal{H}}(\lambda)}{\partial\lambda}\right\rangle_e\\&=\frac{\partial}{\partial\beta}\left[\beta\left\langle\frac{\partial\hat{\mathcal{H}}(\lambda)}{\partial\lambda}\right\rangle_e\right],\end{aligned} \tag{7}$$

and its integral form is

$$\beta\left\langle\frac{\partial\hat{\mathcal{H}}}{\partial\lambda}\right\rangle_e=\frac{\partial}{\partial\lambda}\int d\beta\bar{E}. \tag{8}$$

3 Quantization of Mesoscopic LC Circuits Including Complicated Coupling

Consider mesoscopic LC circuits including complicated coupling (see Fig. 1), where L_l ($l=1,2$) and C_l is, respectively, the inductance and the capacitance of the lth branch circuit, and L_c, and C_c are, respectively, coupling inductance, and coupling capacitance. Here, we have supposed that at the time $t=0$ the system is excitated by an impulse source (the switch time $\tau\to 0$). When the branch circuit charge q_l is referred to as the generalized coordinate, the potential energy of the system is

$$\mathcal{V}=\sum_{l=1}^{2}\frac{q_l^2}{2C_l}+\frac{1}{2C_c}(q_1-q_2)^2, \tag{9}$$

and the kinetic energy corresponding to $\dot{q}_l$ is

$$\mathcal{T} = \sum_{l=1}^{2} \frac{1}{2} L_l \dot{q}_l^2 + \frac{1}{2} L_c (\dot{q}_1 - \dot{q}_2)^2 + M \dot{q}_1 \dot{q}_2, \tag{10}$$

where M is the mutual-inductance between L_1 and L_2. So the Lagrangian function of the system is

$$\mathcal{L} = \mathcal{T} - \mathcal{V}, \tag{11}$$

from which we obtain the generalized momenta

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = (L_1 + L_c)\dot{q}_1 + (M - L_c)\dot{q}_2, \tag{12}$$

$$p_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = (L_2 + L_c)\dot{q}_2 + (M - L_c)\dot{q}_1, \tag{13}$$

which imply p_l and q_l are a pair of canonical conjugate variables. Thus, the Hamiltonian of the system is

$$\begin{aligned} \mathcal{H} &= \sum_{l=1}^{2} p_l \dot{q}_l - \mathcal{L} \\ &= \sum_{l=1}^{2} \frac{1}{2} \left[\frac{1}{(L_l + L_c)\gamma} p_l^2 + (\frac{1}{C_l} + \frac{1}{C}) q_l^2 \right] \\ &\quad - \frac{M - L_c}{\gamma} \prod_{l=1}^{2} \frac{p_l}{(L_l + L_c)} - \frac{1}{C} q_1 q_2, \end{aligned} \tag{14}$$

where $\gamma = 1 - \frac{(M-L)^2}{(L_1+L)(L_2+L)}$. According to Dirac's standard canonical quantization method [16] and endowing the p_l and q_l with a quantization condition

$$[\hat{q}_l, \hat{p}_l] = i\hbar, \tag{15}$$

we can obtain the Hamilton operator of the system

$$\hat{\mathcal{H}} = \frac{1}{2}\lambda_1 \hat{p}_1^2 + \frac{1}{2}\lambda_2 \hat{p}_2^2 - \lambda_3 \hat{p}_1 \hat{p}_2 + \frac{1}{2}\lambda_4 \hat{q}_1^2 + \frac{1}{2}\lambda_5 \hat{q}_2^2 - \lambda_6 \hat{q}_1 \hat{q}_2, \tag{16}$$

where

$$\begin{aligned} &\lambda_1 = \frac{1}{(L_1 + L_c)\gamma}, \qquad \lambda_2 = \frac{1}{(L_2 + L_c)\gamma}, \qquad \lambda_3 = \prod_{l=1}^{2} \frac{M - L_c}{(L_l + L_c)\gamma}, \\ &\lambda_4 = \frac{1}{C_1} + \frac{1}{C_c}, \qquad \lambda_5 = \frac{1}{C_2} + \frac{1}{C_c}, \qquad \lambda_6 = \frac{1}{C_c}. \end{aligned} \tag{17}$$

4 Thermal Effect of the System by Virtue of GHFT

For the Hamilton operator given by (16), introducing the following new bosonic operators

$$\hat{b}_1 = \frac{1}{\sqrt{2\hbar}}\left[\left(\frac{\lambda_4}{\lambda_1}\right)^{1/4}\hat{q}_1 + i\left(\frac{\lambda_1}{\lambda_4}\right)^{1/4}\hat{p}_1\right], \qquad \hat{b}_1^\dagger = \frac{1}{\sqrt{2\hbar}}\left[\left(\frac{\lambda_4}{\lambda_1}\right)^{1/4}\hat{q}_1 - i\left(\frac{\lambda_1}{\lambda_4}\right)^{1/4}\hat{p}_1\right],$$
$$\hat{b}_2 = \frac{1}{\sqrt{2\hbar}}\left[\left(\frac{\lambda_5}{\lambda_2}\right)^{1/4}\hat{q}_2 + i\left(\frac{\lambda_2}{\lambda_5}\right)^{1/4}\hat{p}_2\right], \qquad \hat{b}_2^\dagger = \frac{1}{\sqrt{2\hbar}}\left[\left(\frac{\lambda_5}{\lambda_2}\right)^{1/4}\hat{q}_2 - i\left(\frac{\lambda_2}{\lambda_5}\right)^{1/4}\hat{p}_2\right], \tag{18}$$

we have

$$\hat{\mathcal{H}} = \omega_1'\left(\hat{b}_1^\dagger\hat{b}_1 + \frac{1}{2}\right) + \omega_2'\left(\hat{b}_2^\dagger\hat{b}_2 + \frac{1}{2}\right) + g_1(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_1\hat{b}_2^\dagger) + g_2(\hat{b}_1^\dagger\hat{b}_2^\dagger + \hat{b}_1\hat{b}_2), \tag{19}$$

where

$$[\hat{b}_l, \hat{b}_{l'}^\dagger] = \delta_{ll'}, \qquad [\hat{b}_1, \hat{b}_2^\dagger] = [\hat{b}_1^\dagger, \hat{b}_2] = [\hat{b}_1, \hat{b}_2] = [\hat{b}_1^\dagger, \hat{b}_2^\dagger] = 0, \tag{20}$$

$$\begin{aligned} \omega_1' &= \hbar\sqrt{\lambda_1\lambda_4}, \qquad \omega_2' = \hbar\sqrt{\lambda_2\lambda_5}, \\ g_1 &= -\frac{\hbar}{2}\left(\lambda_3\sqrt{\frac{\lambda_4\lambda_5}{\lambda_1\lambda_2}} + \lambda_6\sqrt{\frac{\lambda_1\lambda_2}{\lambda_4\lambda_5}}\right), \\ g_2 &= \frac{\hbar}{2}\left(\lambda_3\sqrt{\frac{\lambda_4\lambda_5}{\lambda_1\lambda_2}} - \lambda_6\sqrt{\frac{\lambda_1\lambda_2}{\lambda_4\lambda_5}}\right). \end{aligned} \tag{21}$$

The Hamilton operator given by (19) is a formal resemblance with one that describes the biphotonic process. It includes four terms. The former two terms describe the free field process. The latter two terms describe the interacting process in which $g_1(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_1\hat{b}_2^\dagger)$ represents real-photons process and $g_2(\hat{b}_1^\dagger\hat{b}_2^\dagger + \hat{b}_1\hat{b}_2)$, the virtual-photons process. So we can investigate the mesoscopic circuit including complicated coupling by simulating it with a model in quantum optics.

We calculate the ensemble average energy $\langle\hat{\mathcal{H}}\rangle_e$ by employing GHFT. Because $\langle\hat{\mathcal{H}}\rangle_e$ is dependent on ω_1', ω_2', g_1 and g_2, substituting (19) into (7) we obtain

$$\begin{aligned} \frac{\partial\langle\hat{\mathcal{H}}\rangle_e}{\partial\omega_1'} &= \left\langle(1 + \beta\langle\hat{\mathcal{H}}\rangle_e - \beta\hat{\mathcal{H}})\left(\hat{b}_1^\dagger\hat{b}_1 + \frac{1}{2}\right)\right\rangle_e, \\ \frac{\partial\langle\hat{\mathcal{H}}\rangle_e}{\partial\omega_2'} &= \left\langle(1 + \beta\langle\hat{\mathcal{H}}\rangle_e - \beta\hat{\mathcal{H}})\left(\hat{b}_2^\dagger\hat{b}_2 + \frac{1}{2}\right)\right\rangle_e, \\ \frac{\partial\langle\hat{\mathcal{H}}\rangle_e}{\partial g_1} &= \langle(1 + \beta\langle\hat{\mathcal{H}}\rangle_e - \beta\hat{\mathcal{H}})(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_1\hat{b}_2^\dagger)\rangle_e, \\ \frac{\partial\langle\hat{\mathcal{H}}\rangle_e}{\partial g_2} &= \langle(1 + \beta\langle\hat{\mathcal{H}}\rangle_e - \beta\hat{\mathcal{H}})(\hat{b}_1^\dagger\hat{b}_2^\dagger + \hat{b}_1\hat{b}_2)\rangle_e. \end{aligned} \tag{22}$$

In order to solve $\langle\hat{\mathcal{H}}\rangle_e$, we expect to search for an operator Λ so that $[\Lambda, \hat{\mathcal{H}}]$ emerges the four items: $\hat{b}_1^\dagger\hat{b}_1$, $\hat{b}_2^\dagger\hat{b}_2^\dagger$, $(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_1\hat{b}_2^\dagger)$ and $(\hat{b}_1^\dagger\hat{b}_2^\dagger + \hat{b}_1\hat{b}_2)$. Comparing the commutation rejections

$$\left[\frac{g_2}{2}\left(\frac{\hat{b}_1^2-\hat{b}_1^{\dagger 2}}{\omega_1'}-\frac{\hat{b}_2^2-\hat{b}_2^{\dagger 2}}{\omega_2'}\right),\hat{\mathcal{H}}\right]$$
$$= g_2(\hat{b}_1^2+\hat{b}_1^{\dagger 2}-\hat{b}_2^2-\hat{b}_2^{\dagger 2})$$
$$+ g_2\left(\frac{1}{\omega_1'}-\frac{1}{\omega_2'}\right)[g_1(\hat{b}_1^\dagger\hat{b}_2^\dagger+\hat{b}_1\hat{b}_2)+g_2(\hat{b}_1^\dagger\hat{b}_2+\hat{b}_1\hat{b}_2^\dagger)], \tag{23}$$

and

$$[\hat{b}_1^\dagger\hat{b}_2-\hat{b}_1\hat{b}_2^\dagger,\hat{\mathcal{H}}] = 2g_1(\hat{b}_1^\dagger\hat{b}_1^\dagger-\hat{b}_2\hat{b}_2)-(\omega_1'-\omega_2')(\hat{b}_1^\dagger\hat{b}_2+\hat{b}_1\hat{b}_2^\dagger)$$
$$+ g_2(\hat{b}_1^{\dagger 2}+\hat{b}_1^2-\hat{b}_2^2-\hat{b}_2^\dagger), \tag{24}$$

leads to

$$[\Lambda,\hat{\mathcal{H}}] = 2g_1(\hat{b}_1^\dagger\hat{b}_1^\dagger-\hat{b}_2\hat{b}_2)-\left(\omega_1'-\omega_2'+\frac{g_2^2}{\omega_1'}-\frac{g_2^2}{\omega_2'}\right)(\hat{b}_1^\dagger\hat{b}_2+\hat{b}_1\hat{b}_2^\dagger)$$
$$- g_1g_2\left(\frac{1}{\omega_1'}-\frac{1}{\omega_2'}\right)(\hat{b}_1^\dagger\hat{b}_2^\dagger+\hat{b}_1\hat{b}_2), \tag{25}$$

where we have defined

$$\Lambda \equiv (\hat{b}_1^\dagger\hat{b}_2-\hat{b}_1\hat{b}_2^\dagger)-\frac{g_2}{2}\left(\frac{\hat{b}_1^2-\hat{b}_1^{\dagger 2}}{\omega_1'}-\frac{\hat{b}_2^2-\hat{b}_2^{\dagger 2}}{\omega_2'}\right). \tag{26}$$

Because $|\Theta_n\rangle$ is the eigenstate of $\hat{\mathcal{H}}$, the following relation is obtained easily

$$\langle\Theta_n|[\Lambda,\hat{\mathcal{H}}]|\Theta_n\rangle = 0, \tag{27}$$

which leads to

$$\langle(1+\beta\langle\hat{\mathcal{H}}\rangle_e-\beta\hat{\mathcal{H}})[\Lambda,\hat{\mathcal{H}}]\rangle_e$$
$$= \mathrm{Tr}\{e^{-\beta\hat{\mathcal{H}}}(1+\beta\langle\hat{\mathcal{H}}\rangle_e-\beta\hat{\mathcal{H}})[\Lambda,\hat{\mathcal{H}}]\}/\,\mathrm{Tr}(e^{-\beta\hat{\mathcal{H}}})$$
$$= \frac{1}{\mathrm{Tr}(e^{-\beta\hat{\mathcal{H}}})}\sum_n e^{-\beta E_n}(1+\beta\langle\hat{\mathcal{H}}\rangle_e-\beta\hat{\mathcal{H}})\langle\Theta_n|[\Lambda,\hat{\mathcal{H}}]|\Theta_n\rangle$$
$$= \frac{1}{\mathrm{Tr}(e^{-\beta\hat{\mathcal{H}}})}\sum_n e^{-\beta E_n}(1+\beta\langle\hat{\mathcal{H}}\rangle_e-\beta\hat{\mathcal{H}})[2g_1\langle\Theta_n|(\hat{b}_1^\dagger\hat{b}_1^\dagger-\hat{b}_2\hat{b}_2)|\Theta_n\rangle$$
$$-\left(\omega_1'-\omega_2'+\frac{g_2^2}{\omega_1'}-\frac{g_2^2}{\omega_2'}\right)\langle\Theta_n|(\hat{b}_1^\dagger\hat{b}_2+\hat{b}_1\hat{b}_2^\dagger)|\Theta_n\rangle - g_1g_2$$
$$\times\left(\frac{1}{\omega_1'}-\frac{1}{\omega_2'}\right)\langle\Theta_n|(\hat{b}_1^\dagger\hat{b}_2^\dagger+\hat{b}_1\hat{b}_2)|\Theta_n\rangle] = 0. \tag{28}$$

Comparing (28) with (22) yields

$$2g_1\left(\frac{\partial\langle\hat{\mathcal{H}}\rangle_e}{\partial\omega_1'}-\frac{\partial\langle\hat{\mathcal{H}}\rangle_e}{\partial\omega_2'}\right)-(\omega_1'-\omega_2')\left(1-\frac{g_2^2}{\omega_1'\omega_2'}\right)\frac{\partial\langle\hat{\mathcal{H}}\rangle_e}{\partial g_1}+\frac{g_1g_2(\omega_1'-\omega_2')}{\omega_1'\omega_2'}\frac{\partial\langle\hat{\mathcal{H}}\rangle_e}{\partial g_2}=0, \tag{29}$$

which is a 1-order partial differential equation about $\langle\hat{\mathcal{H}}\rangle_e$ and can be solved by using the method of Characteristics [17]. And the corresponding characteristic equation is

$$\frac{d\omega_1'}{2g_1} = -\frac{d\omega_2'}{2g_1} = -\frac{dg_1}{(\omega_1'-\omega_2')(1-\frac{g_2^2}{\omega_1'\omega_2'})} = \frac{\omega_1'\omega_2' dg_2}{g_1 g_2(\omega_1'-\omega_2')}, \tag{30}$$

from which we obtain

$$d\omega_1' = -d\omega_2', \qquad g_1 dg_1 = \left(g_2 - \frac{\omega_1'\omega_2'}{g_2}\right)dg_2, \tag{31}$$

and

$$\begin{aligned}\frac{d\omega_2'}{2g_1} &= \frac{dg_2}{g_1 g_2(\frac{1}{\omega_1'}-\frac{1}{\omega_2'})}\\ &= \frac{\omega_1' d\omega_1' + \omega_2' d\omega_2' + 2g_1 dg_1 - 2g_2 dg_2}{0}.\end{aligned} \tag{32}$$

Equation (32) implies

$$d(\omega_1'^2 + \omega_2'^2 + 2g_1^2 - 2g_2^2) = 0. \tag{33}$$

Integrating the above three characteristic equations yields

$$\omega_1' + \omega_2' = \mathcal{C}_1, \qquad \omega_1'\omega_2' g_2^2 = \mathcal{C}_2, \qquad \omega_1'^2 + \omega_2'^2 + 2g_1^2 - 2g_2^2 = \mathcal{C}_3', \tag{34}$$

where $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3'$ are constants. For convenience, we suppose $\mathcal{C}_3' = \mathcal{C}_1^2 + 2\mathcal{C}_3$ so that

$$-\omega_1'\omega_2' + g_1^2 - g_2^2 = \mathcal{C}_3. \tag{35}$$

Thus according to the method of Characteristics, the general solution of $\langle\hat{\mathcal{H}}\rangle_e$ is

$$\langle\hat{\mathcal{H}}\rangle_e = \mathcal{F}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3). \tag{36}$$

Here, $\mathcal{F}$ is an arbitrary function about $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$, and usually its form is dependent on the initial conditions of the 1-order partial differential equation about $\langle\hat{\mathcal{H}}\rangle_e$. Supposing that $g_2 \longrightarrow 0$ at the initial time, we obtain $\hat{\mathcal{H}} \longrightarrow \omega_1'(\hat{b}_1^\dagger\hat{b}_1 + \frac{1}{2}) + \omega_2'(\hat{b}_2^\dagger\hat{b}_2 + \frac{1}{2}) + g_1(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_1\hat{b}_2^\dagger)$, $\mathcal{C}_2 \longrightarrow 0$ and $\mathcal{C}_3 \longrightarrow -\omega_1'\omega_2' + g_1^2 \equiv \mathcal{C}_{30}$. Construct the following two parameters

$$\begin{aligned}\mathcal{A}' &= \frac{\mathcal{C}_1 + \sqrt{\mathcal{C}_1^2 + 4\mathcal{C}_{30}}}{2} = \frac{\omega_1' + \omega_2' + \sqrt{(\omega_1'-\omega_2')^2 + 4g_1^2}}{2},\\ \mathcal{B}' &= \frac{\mathcal{C}_1 - \sqrt{\mathcal{C}_1^2 + 4\mathcal{C}_{30}}}{2} = \frac{\omega_1' + \omega_2' - \sqrt{(\omega_1'-\omega_2')^2 + 4g_1^2}}{2},\end{aligned} \tag{37}$$

which satisfy $\mathcal{A}' + \mathcal{B}' = \mathcal{C}_1$ and $\mathcal{A}'\mathcal{B}' = -\mathcal{C}_{30}$. Thus we have $\langle\hat{\mathcal{H}}\rangle_e|_{g_2\to 0} = \mathcal{F}'(\mathcal{A}', \mathcal{B}')$. In addition, let $g_1 \longrightarrow 0$, so $\mathcal{A}' \longrightarrow \omega_1'$ and $\mathcal{B}' \longrightarrow \omega_2'$. Moreover in this case $\hat{\mathcal{H}} \longrightarrow \omega_1'(\hat{b}_1^\dagger\hat{b}_1 + \frac{1}{2}) + \omega_2'(\hat{b}_2^\dagger\hat{b}_2 + \frac{1}{2})$, so we obtain

$$\langle\hat{\mathcal{H}}\rangle_e|_{g_1\to 0, g_2\to 0} = \mathcal{F}'(\omega_1', \omega_2') = \mathcal{F}_1'(\omega_1') + \mathcal{F}_1'(\omega_2'), \tag{38}$$

where $\mathcal{F}'_1(\omega'_l)$ $(l=1,2)$ is the energy ensemble average of the single harmonic oscillator, i.e.,

$$\begin{aligned}\mathcal{F}'_1(\omega'_l) &= \frac{\mathrm{Tr}[e^{-\beta\omega'_l(\hat{b}_l^\dagger\hat{b}_l+\frac{1}{2})}\omega'_l(\hat{b}_l^\dagger\hat{b}_l+\frac{1}{2})]}{\mathrm{Tr}[e^{-\beta\omega'_l(\hat{b}_l^\dagger\hat{b}_l+\frac{1}{2})}]} = -\frac{\partial}{\partial\beta}\ln\{\mathrm{Tr}[e^{-\beta\omega'_l(\hat{b}_l^\dagger\hat{b}_l+\frac{1}{2})}]\} \\ &= \frac{1}{2}\omega'_l + \frac{\omega'_l}{e^{\beta\omega'_l}-1} \quad (l=1,2). \end{aligned} \tag{39}$$

Thus we deduce

$$\begin{aligned} &\mathcal{F}'_1(\mathcal{A}') == \frac{1}{2}\mathcal{A}' + \frac{\mathcal{A}'}{e^{\beta\mathcal{A}'}-1}, \qquad \mathcal{F}'_1(\mathcal{B}') = \frac{1}{2}\mathcal{B}' + \frac{\mathcal{B}'}{e^{\beta\mathcal{B}'}-1}, \\ &\langle\hat{\mathcal{H}}\rangle_e|_{g_2\to 0} = \mathcal{F}'(\mathcal{A}',\mathcal{B}') = \mathcal{F}'_1(\mathcal{A}') + \mathcal{F}'_1(\mathcal{B}'). \end{aligned} \tag{40}$$

In order to determine the form of $\mathcal{F}$, introduce

$$\mathcal{A} = \frac{1}{\sqrt{2}}(\mathcal{C}_1^2 + 2\mathcal{C}_3 + \mathcal{K})^{1/2}, \qquad \mathcal{B} = \frac{1}{\sqrt{2}}(\mathcal{C}_1^2 + 2\mathcal{C}_3 - \mathcal{K})^{1/2}, \tag{41}$$

where

$$\mathcal{K} \equiv \sqrt{\mathcal{C}_1^4 + 4\mathcal{C}_1^2\mathcal{C}_3 + 16\mathcal{C}_2} = \sqrt{(\omega_1'^2 - \omega_2'^2)^2 + 4g_1^2(\omega'_1+\omega'_2)^2 - 4g_2^2(\omega'_1-\omega'_2)^2}. \tag{42}$$

When $g_2 \longrightarrow 0$, the following relations can be obtained

$$\begin{aligned} \mathcal{A} &\longrightarrow \frac{1}{2}\big(2\mathcal{C}_1^2 + 4\mathcal{C}_{30} + 2\mathcal{C}_1\sqrt{\mathcal{C}_1^2+4\mathcal{C}_{30}}\big)^{1/2} = \frac{1}{2}\big(\mathcal{C}_1 + \sqrt{\mathcal{C}_1^2+4\mathcal{C}_{30}}\big) = \mathcal{A}', \\ \mathcal{B} &\longrightarrow \frac{1}{2}\big(2\mathcal{C}_1^2 + 4\mathcal{C}_{30} - 2\mathcal{C}_1\sqrt{\mathcal{C}_1^2+4\mathcal{C}_{30}}\big)^{1/2} = \frac{1}{2}\big(\mathcal{C}_1 - \sqrt{\mathcal{C}_1^2+4\mathcal{C}_{30}}\big) = \mathcal{B}'. \end{aligned} \tag{43}$$

Similar to (40), the energy ensemble average of the coupling system can be deduced

$$\langle\hat{\mathcal{H}}\rangle_e \equiv \mathcal{F}(\mathcal{A},\mathcal{B}) = \mathcal{F}_1(\mathcal{A}) + \mathcal{F}_1(\mathcal{B}) = \frac{1}{2}(\mathcal{A}+\mathcal{B}) + \frac{\mathcal{A}}{e^{\beta\mathcal{A}}-1} + \frac{\mathcal{B}}{e^{\beta\mathcal{B}}-1}, \tag{44}$$

which explicitly shows the influence of temperature. Next, we discuss the contribution of each process to the energy ensemble average. From (8) and (44), we have

$$\begin{aligned} &\omega'_1\left\langle\hat{b}_1^\dagger\hat{b}_1 + \frac{1}{2}\right\rangle_e \\ &\quad = \omega'_1\left\langle\frac{\partial\hat{\mathcal{H}}}{\partial\omega'_1}\right\rangle_e = \frac{\omega'_1\partial}{\beta\partial\omega'_1}\int d\beta\,\bar{E} \\ &\quad = \sum_{l,j=0,1(l\neq j)}\left\{\frac{1}{4\mathcal{A}^l\mathcal{B}^j}\omega'_1\coth\left(\frac{\mathcal{A}^l\mathcal{B}^j\beta}{2}\right)\left[\omega'_1 + (-1)^j\frac{1}{\mathcal{K}}(\mathcal{C}_1^2\omega'_1 + 2\mathcal{C}_1\mathcal{C}_3 + 4\omega'_2 g_2^2)\right]\right\}, \end{aligned} \tag{45}$$

$$\omega_2'\left\langle \hat{b}_2^\dagger \hat{b}_2 + \frac{1}{2}\right\rangle_e$$
$$= \sum_{l,j=0,1(l\neq j)} \left\{ \frac{1}{4\mathcal{A}^l\mathcal{B}^j}\omega_2' \coth\left(\frac{\mathcal{A}^l\mathcal{B}^j\beta}{2}\right)\left[\omega_2' + (-1)^j\frac{1}{\mathcal{K}}(\mathcal{C}_1^2\omega_2' + 2\mathcal{C}_1\mathcal{C}_3 + 4\omega_1' g_2^2)\right]\right\}, \tag{46}$$

$$g_1\langle \hat{b}_1^\dagger \hat{b}_2 + \hat{b}_1 \hat{b}_2^\dagger\rangle_e = \sum_{l,j=0,1(l\neq j)} \left\{ \frac{g_1^2}{2\mathcal{A}^l\mathcal{B}^j} \coth\left(\frac{\mathcal{A}^l\mathcal{B}^j\beta}{2}\right)\left[1 + (-1)^j\frac{1}{\mathcal{K}}\mathcal{C}_1^2\right]\right\}, \tag{47}$$

$$g_2\langle \hat{b}_1^\dagger \hat{b}_2^\dagger + \hat{b}_1 \hat{b}_2\rangle_e = \sum_{l,j=0,1(l\neq j)} \left\{ \frac{g_2^2}{2\mathcal{A}^l\mathcal{B}^j} \coth\left(\frac{\mathcal{A}^l\mathcal{B}^j\beta}{2}\right)\left[-1 + (-1)^j\frac{1}{\mathcal{K}}(-\mathcal{C}_1^2 + 4\omega_1'\omega_2'\right]\right\}. \tag{48}$$

Thus by using GHFT and the method of Characteristics, we obtain the energy ensemble average and the contribution of each process to it, as will be helpful to analysing the mesoscopic circuit and mastering furtherly its law.

In summary, using GHFT and the method of Characteristics, we obtain the energy ensemble average of the mesoscopic LC circuits including complicated coupling and the contribution of each process to it. The effect of temperature can be saw distinctly. We believe that the above conclusions and the methods employed will be helpful to analysing the mesoscopic circuit.

Acknowledgement We thank Prof. Hong-Yi Fan for his helpful discussions.

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